UNIVERSITY OF WISCONSIN–MADISON

CS639:
Data Management for Data Science

Lecture 13: Statistical Inference

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Announcements

- We have graded most of the exams
  - We will be announcing grades by Wednesday
- Next PA will come after the spring break

Where are we?

- Covered data management systems (how to manipulate data)

- For this part of the class we will cover **modeling** and **statistical analysis** (how to obtain insights)

- In the last part of the class we will discuss how to communicate our findings (how to visualize findings)

Today’s Lecture

1. Intro to Statistical Inference
2. Central Limit Theorem and Statistics of Distributions
3. Confidence Intervals
4. Hypothesis Testing

1. Statistical Inference

Statistical Inference

- Statistical inference: The process of making guesses about the truth from sample data.

Sample statistics

$$\hat{\mu} = \bar{X}_n = \frac{\sum_{i=1}^{n} x}{n}$$

$$\hat{\sigma}^2 = s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{X}_n)^2}{n-1}$$

<u>Truth</u> (not observable)

Population parameters

$$\mu = \frac{\sum_{i=1}^{N} x}{N} \qquad \sigma^2 = \frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N}$$

<u>Sample</u> (observation)

*hat notation ^ is often used to indicate "estitmate"

Make guesses about the whole population

Statistics vs. Parameters

- **Sample Statistic** – any summary measure calculated from data; e.g., could be a mean, a difference in means or proportions, an odds ratio, or a correlation coefficient
  - E.g., the mean vitamin D level in a sample of 100 people is 63 nmol/L
  - E.g., the correlation coefficient between vitamin D and cognitive function in the sample of 100 people is 0.15

- **Population parameter** – the true value/true effect in the entire population of interest
  - E.g., the true mean vitamin D in all middle-aged humans is 62 nmol/L
  - E.g., the true correlation between vitamin D and cognitive function in all middle-aged humans is 0.15

Examples of Sample Statistics

Single population mean

Single population proportion

Difference in means (t-test)

Difference in proportions (Z-test)

Odds ratio/risk ratio

Correlation coefficient

Regression coefficient

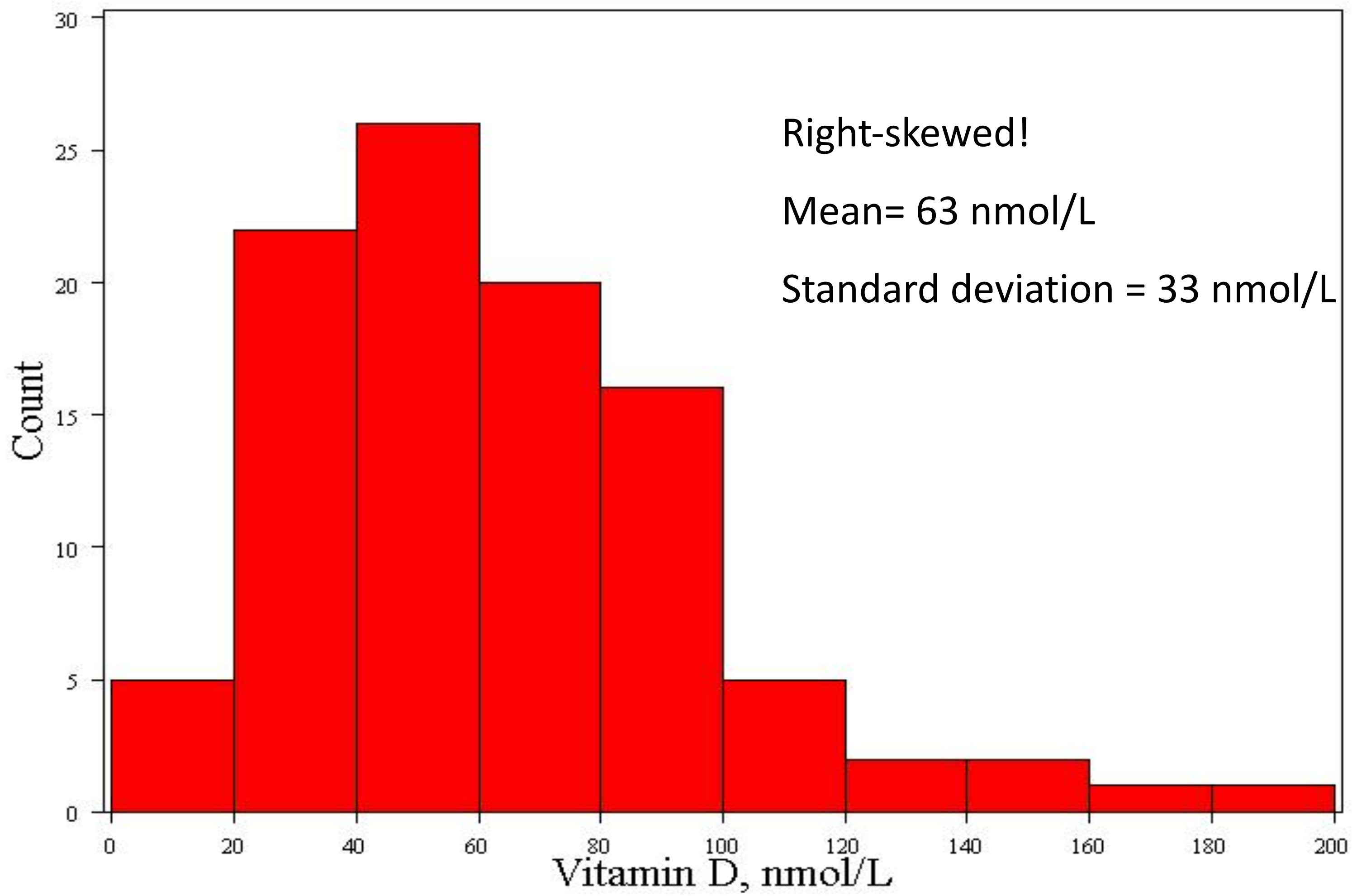
…

Example 1: cognitive function and vitamin D

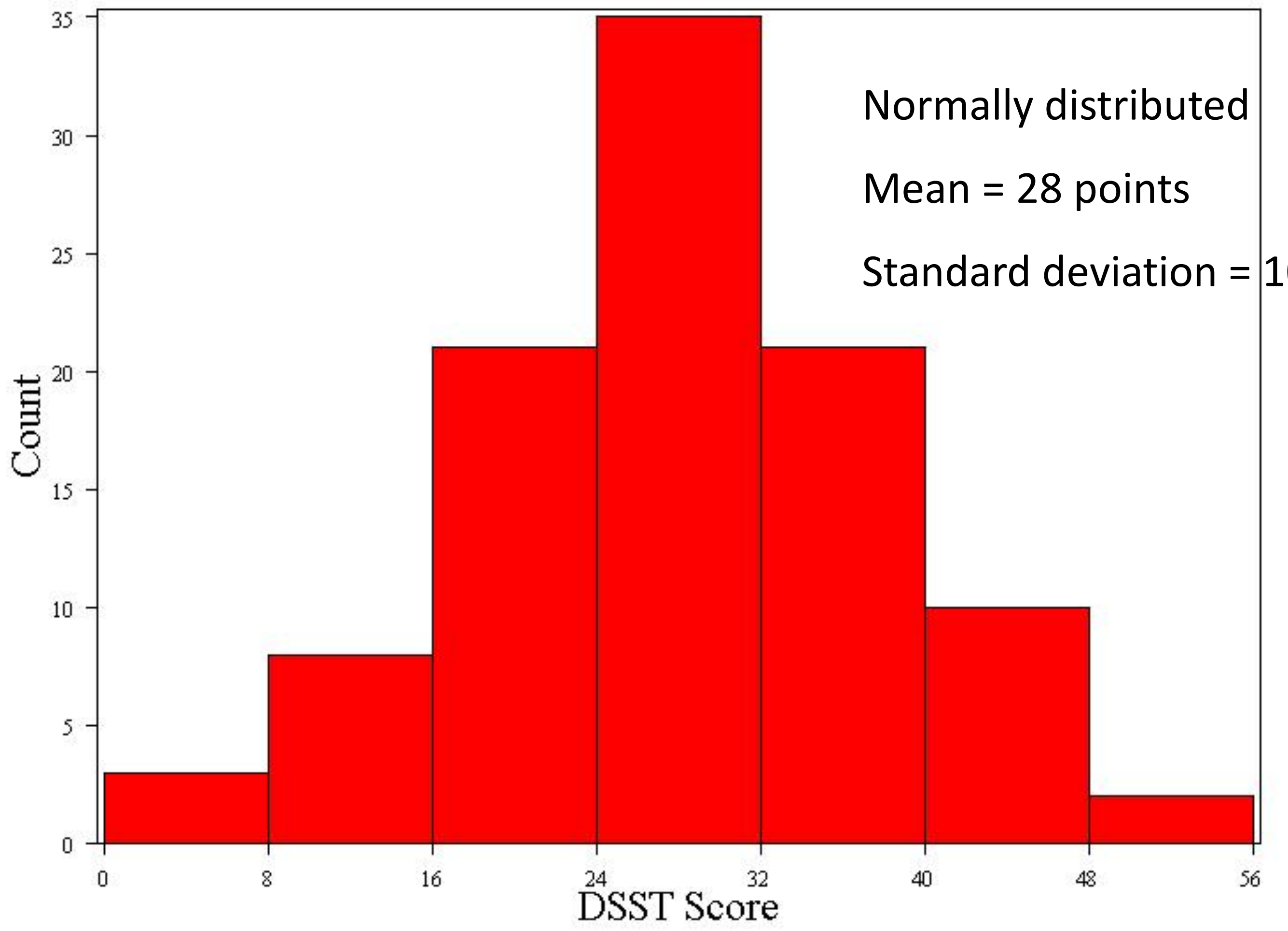
- Hypothetical data loosely based on [1]; cross-sectional study of 100 middle-aged and older European men.
- Estimation: What is the average serum vitamin D in middle-aged and older European men?
  - Sample statistic: mean vitamin D levels
- Hypothesis testing: Are vitamin D levels and cognitive function correlated?
  - Sample statistic: correlation coefficient between vitamin D and cognitive function, measured by the Digit Symbol Substitution Test (DSST).

1. Lee DM, Tajar A, Ulubaev A, et al. Association between 25-hydroxyvitamin D levels and cognitive performance in middle-aged and older European men. J Neurol Neurosurg Psychiatry. 2009 Jul;80(7):722-9.

Distribution of a trait: vitamin D



Distribution of a trait: DSST



Normally distributed

Mean = 28 points

Standard deviation = 10 points

Distribution of a statistic

- Statistics follow distributions too…
- *But the distribution of a statistic is a theoretical construct.*
- Statisticians ask a thought experiment: how much would the value of the statistic fluctuate if one could repeat a particular study over and over again with different samples of the same size?
- By answering this question, statisticians are able to pinpoint exactly how much uncertainty is associated with a given statistic.

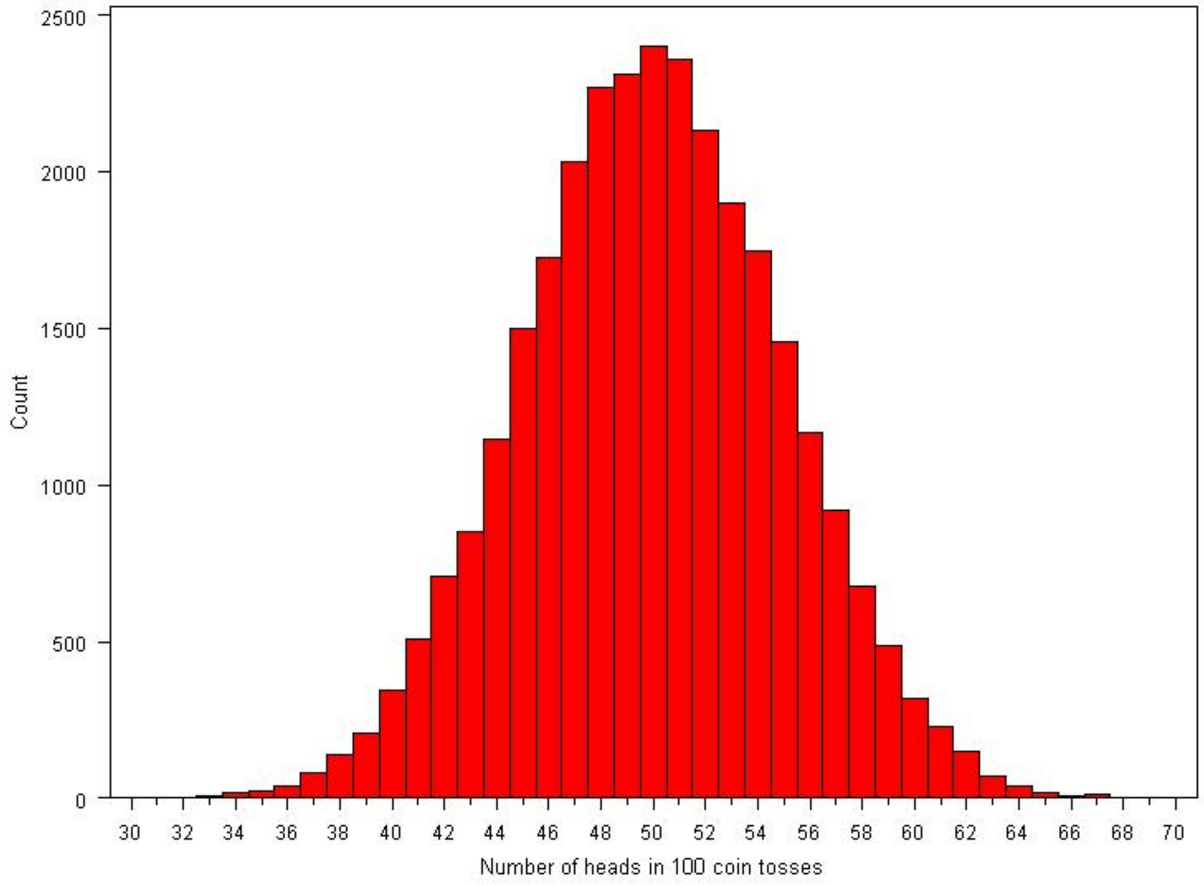
Distribution of a statistic

- Two approaches to determine the distribution of a statistic:
  - 1. Computer simulation
    - Repeat the experiment over and over again virtually!
    - More intuitive; can directly observe the behavior of statistics.
  - 2. Mathematical theory
    - Proofs and formulas!
    - More practical; use formulas to solve problems.

Example of computer simulation

- How many heads come up in 100 coin tosses?
- Flip coins virtually
  - Flip a coin 100 times; count the number of heads.
  - Repeat this over and over again a large number of times (we’ll try 30,000 repeats!)
  - Plot the 30,000 results.

Coin tosses



Conclusions:

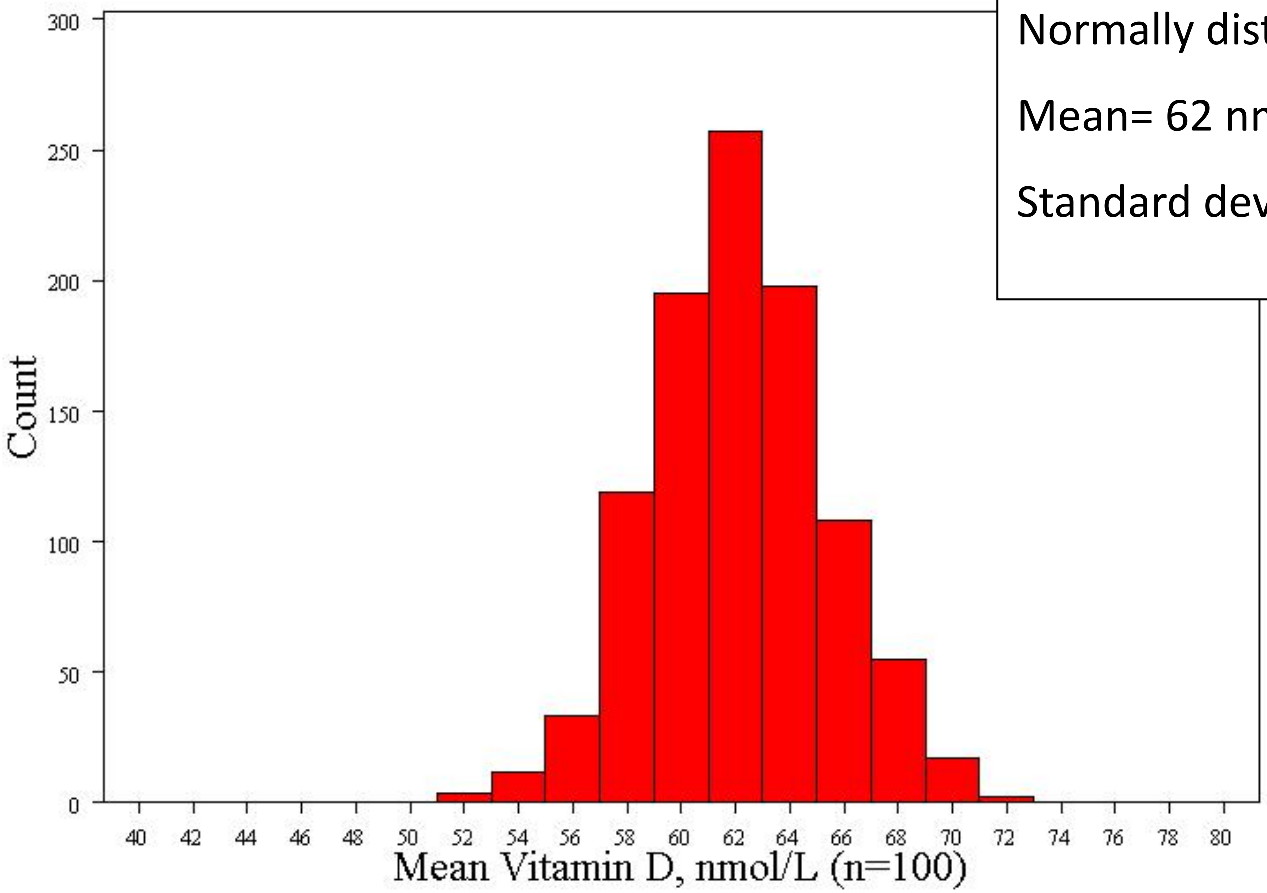
We usually get between 40 and 60 heads when we flip a coin 100 times.

It's extremely unlikely that we will get 30 heads or 70 heads (didn't happen in 30,000 experiments!).

Distribution of the sample mean, computer simulation

- 1. Specify the underlying distribution of vitamin D in all European men aged 40 to 79.
  - Right-skewed
  - Standard deviation = 33 nmol/L
  - True mean = 62 nmol/L (this is arbitrary; does not affect the distribution)
- 2. Select a random sample of 100 virtual men from the population.
- 3. Calculate the mean vitamin D for the sample.
- 4. Repeat steps (2) and (3) a large number of times (say 1000 times).
- 5. Explore the distribution of the 1000 means.

Distribution of mean vitamin D (a sample statistic)



Normally distributed! Surprise!

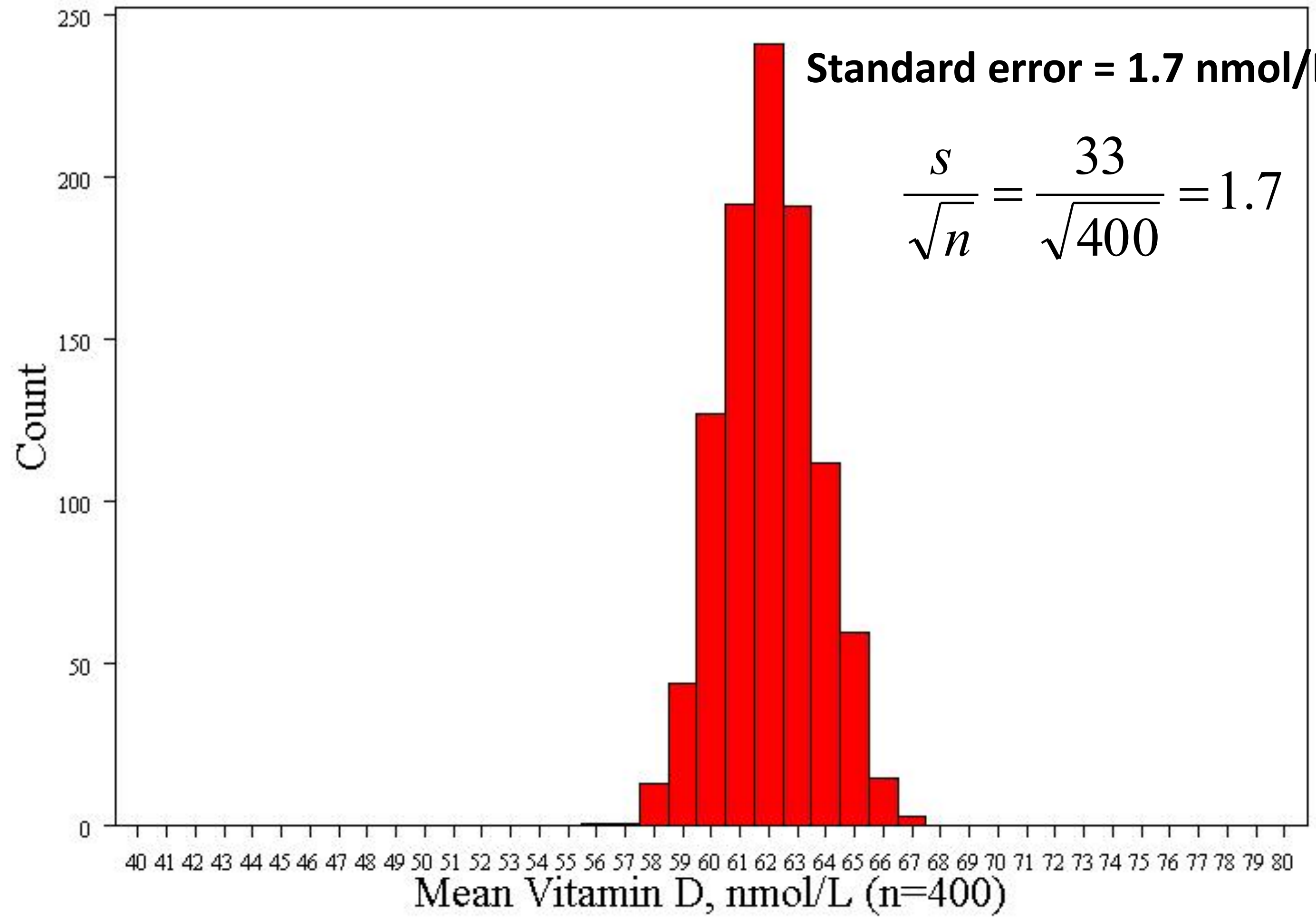
Mean= 62 nmol/L (the true mean)

Standard deviation = 3.3 nmol/L

Distribution of mean vitamin D (a sample statistic)

- Normally distributed (even though the trait is right-skewed!)
- Mean = true mean
- Standard deviation = 3.3 nmol/L
  - <u>The standard deviation of a statistic is called a standard error</u>
  - The standard error of a mean = $\frac{s}{\sqrt{n}}$

If we increase the sample size to n=400

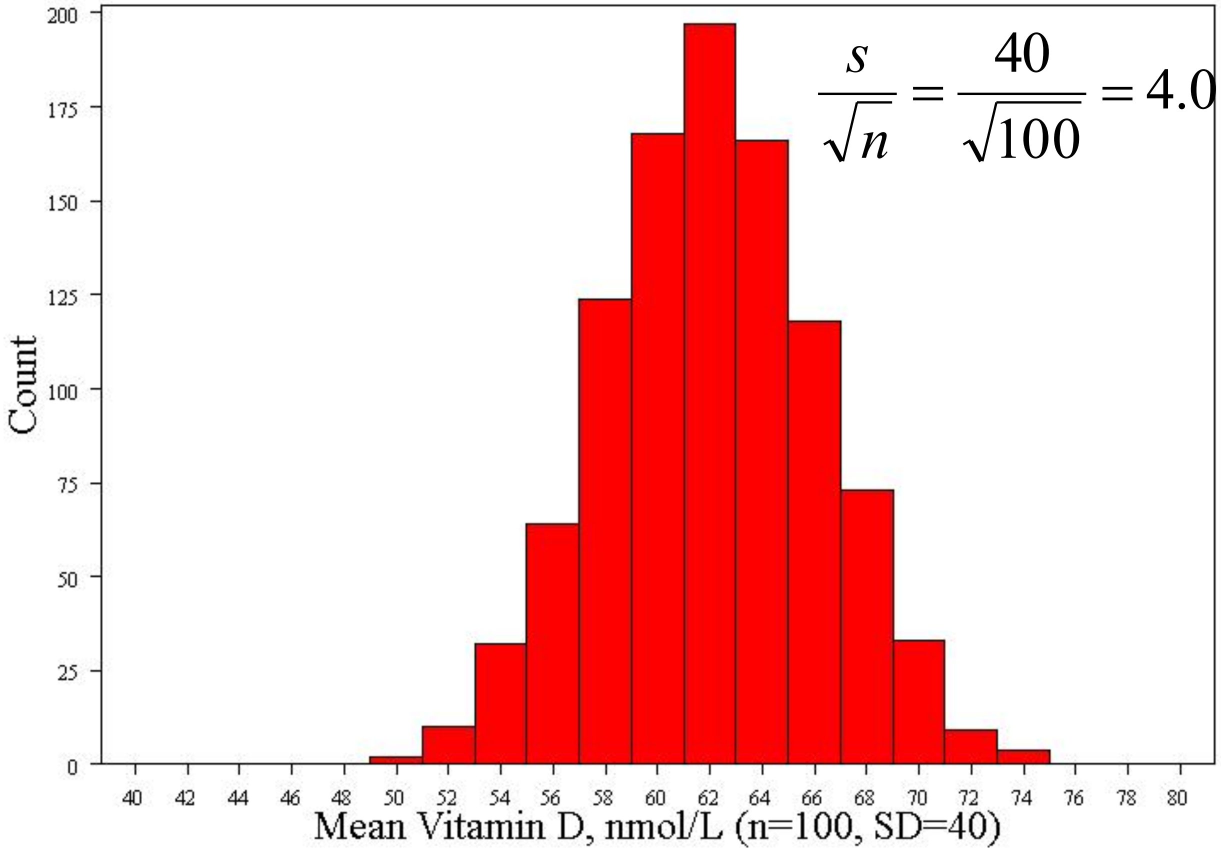


Standard error = 1.7 nmol/L

$$\frac{s}{\sqrt{n}} = \frac{33}{\sqrt{400}} = 1.7$$

If we increase the variability of vitamin D (the trait) to SD = 40

Standard error = 4.0 nmol/L



$$\frac{s}{\sqrt{n}} = \frac{40}{\sqrt{100}} = 4.0$$

2. CLT and Statistics of Distributions

The Central Limit Theorem

If all possible random samples, each of size *n*, are taken from any population with a mean μ and a standard deviation σ, the sampling distribution of the sample means (averages) will:

1. have mean: $\boxed{\mu_{\bar{x}} = \mu}$

2. have standard deviation: $\boxed{\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}}$

3. be approximately normally distributed regardless of the shape of the parent population (normality improves with larger *n*).

Symbol Check

$\mu_{\bar{x}}$ The mean of the sample means.

$\sigma_{\bar{x}}$ The standard deviation of the sample means. *Also called "the standard error of the mean."*

Proof

If *X* is a random variable from any distribution with known mean, E(*x*), and variance, Var(*x*), then the expected value and variance of the average of *n* observations of X is:

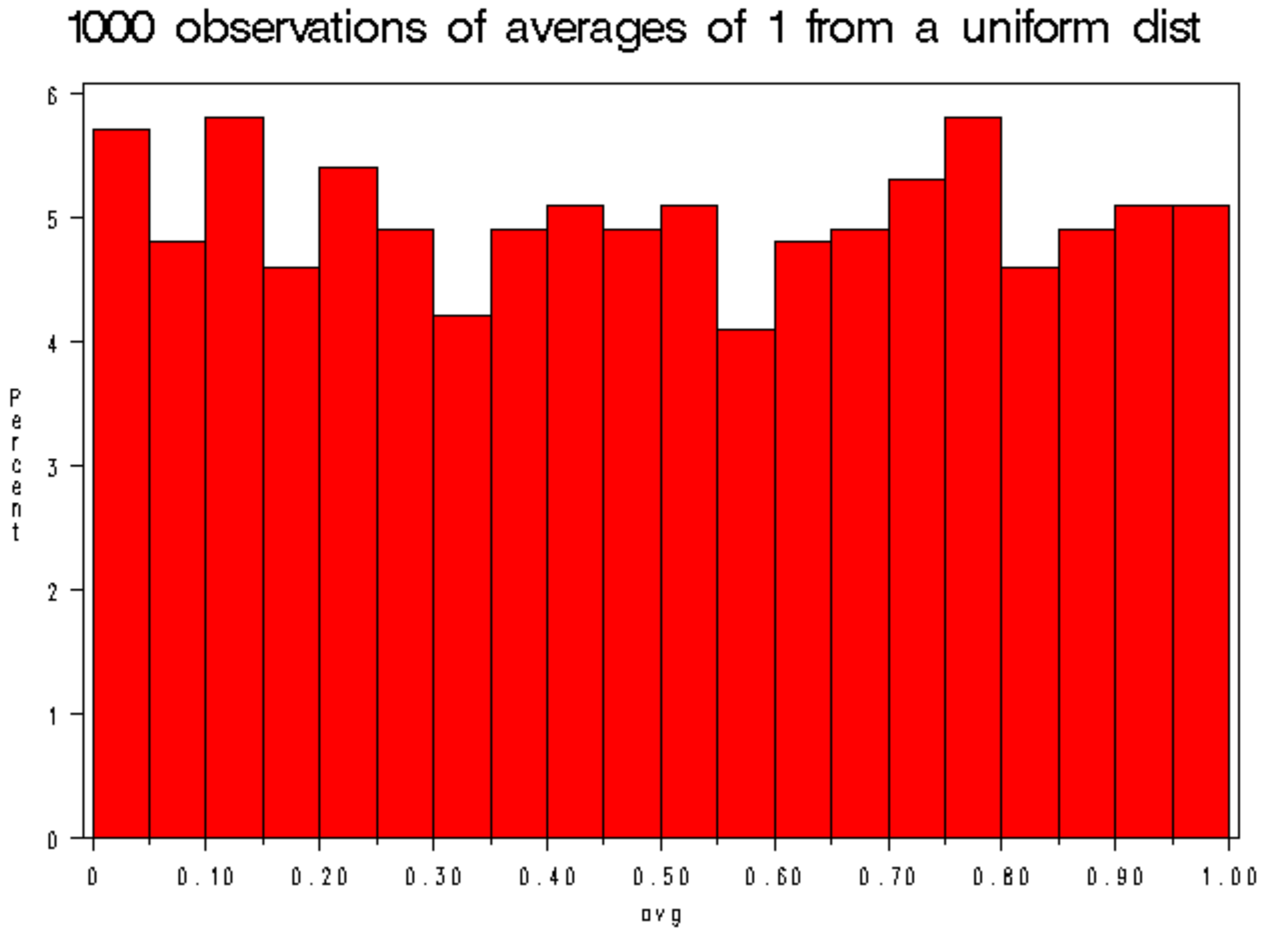
$$E(\bar{X}_n) = E\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) = \frac{\sum_{i=1}^{n} E(x)}{n} = \frac{nE(x)}{n} = E(x)$$

$$Var(\bar{X}_n) = Var\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) = \frac{\sum_{i=1}^{n} Var(x)}{n^2} = \frac{nVar(x)}{n^2} = \frac{Var(x)}{n}$$

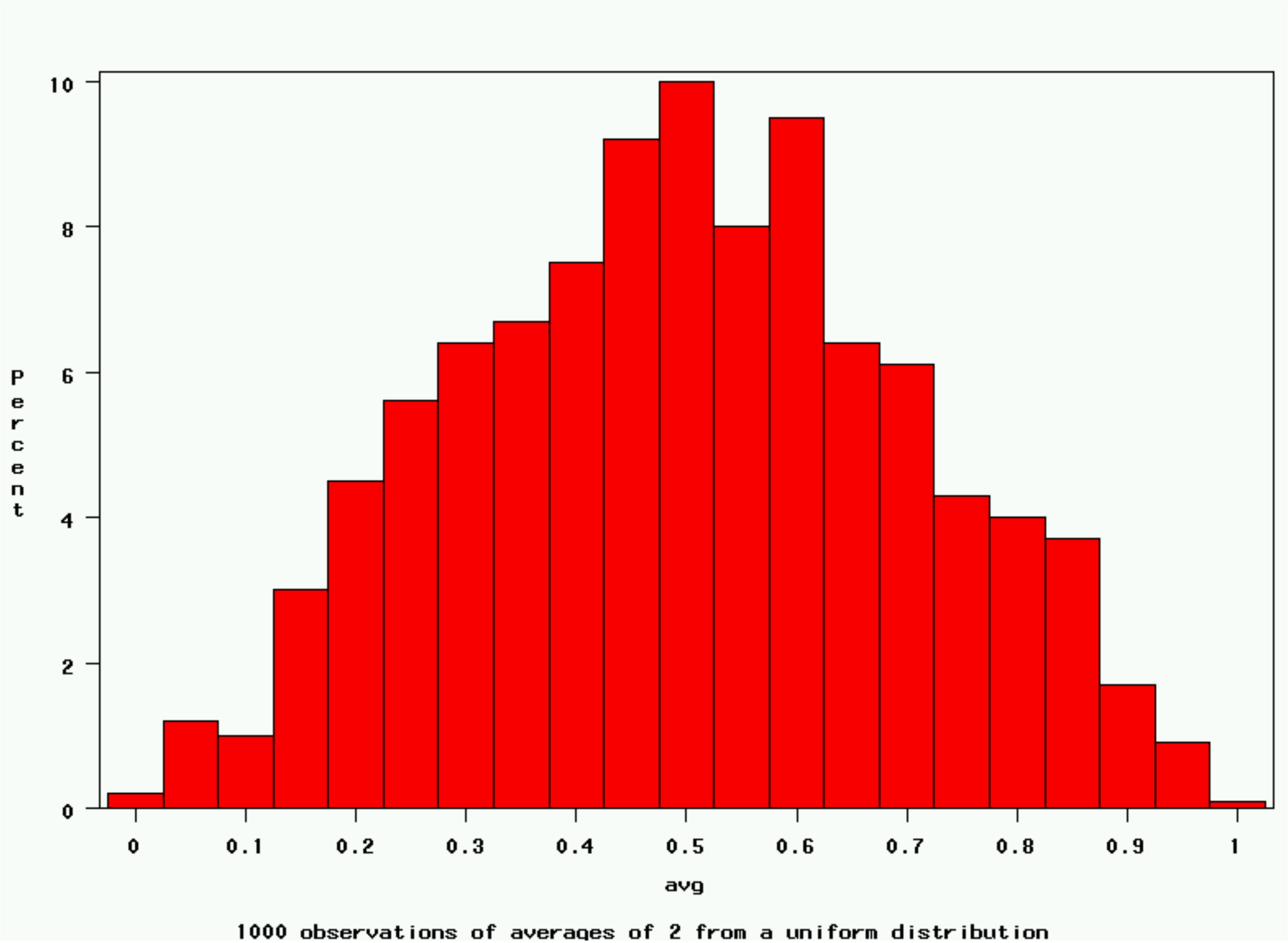
Computer simulation of the CLT

1. Pick any probability distribution and specify a mean and standard deviation.
2. Tell the computer to randomly generate 1000 observations from that probability distributions
   - E.g., the computer is more likely to spit out values with high probabilities
3. Plot the “observed” values in a histogram.
4. Next, tell the computer to randomly generate 1000 averages-of-2 (randomly pick 2 and take their average) from that probability distribution. Plot “observed” averages in histograms.
5. Repeat for averages-of-10, and averages-of-100.

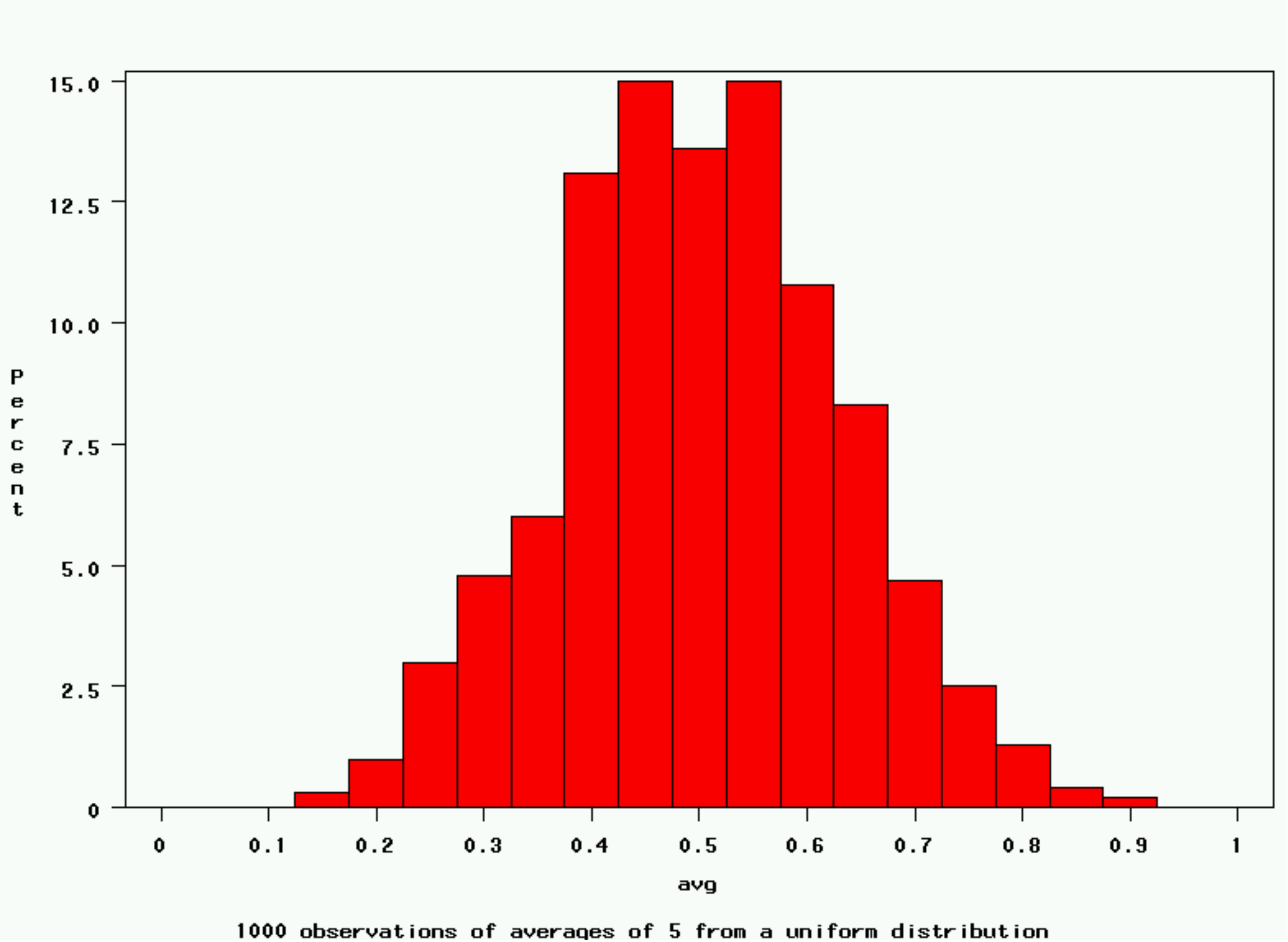
Uniform on [0,1]: average of 1 (original distribution)



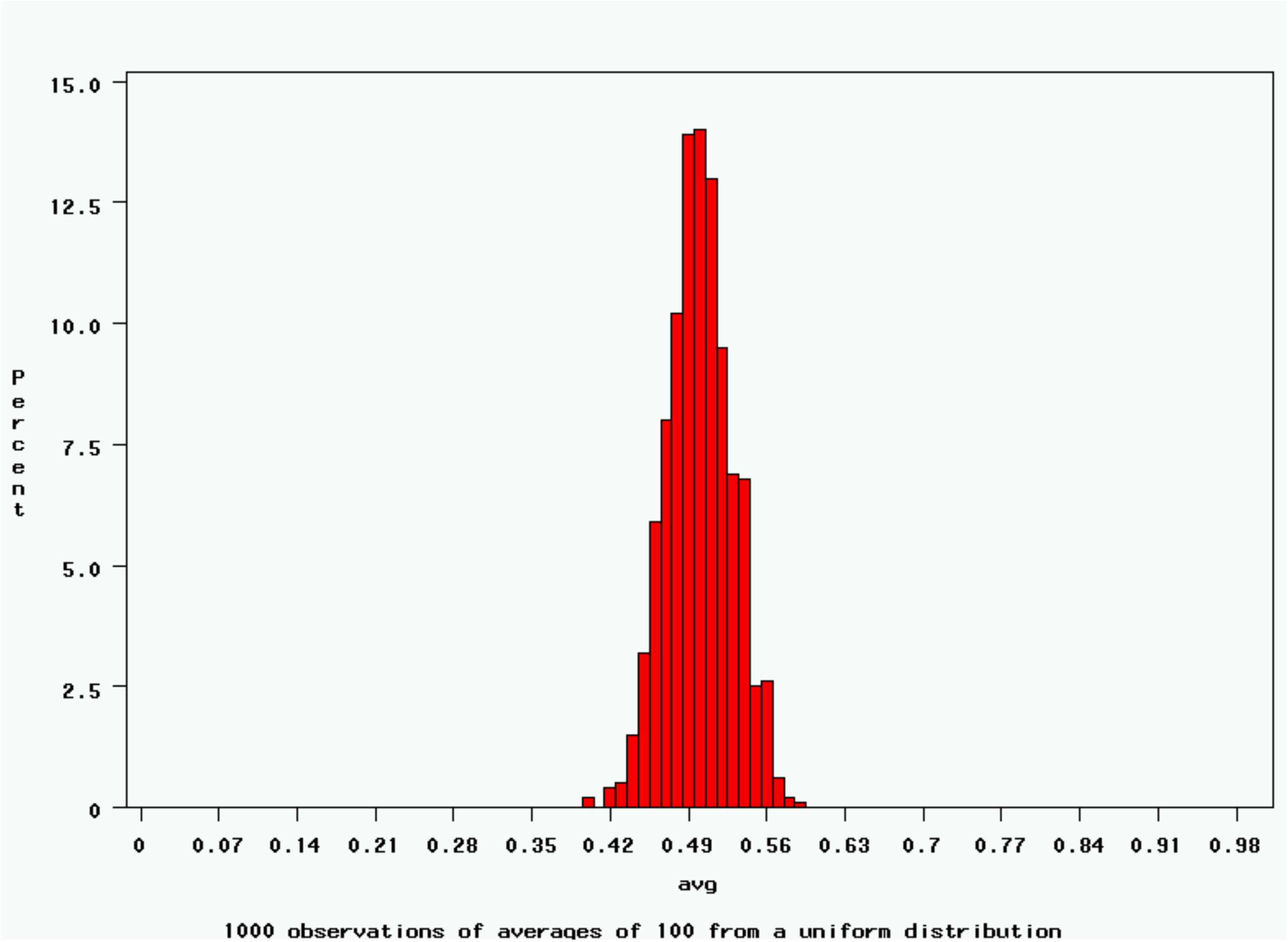
Uniform: 1000 averages of 2



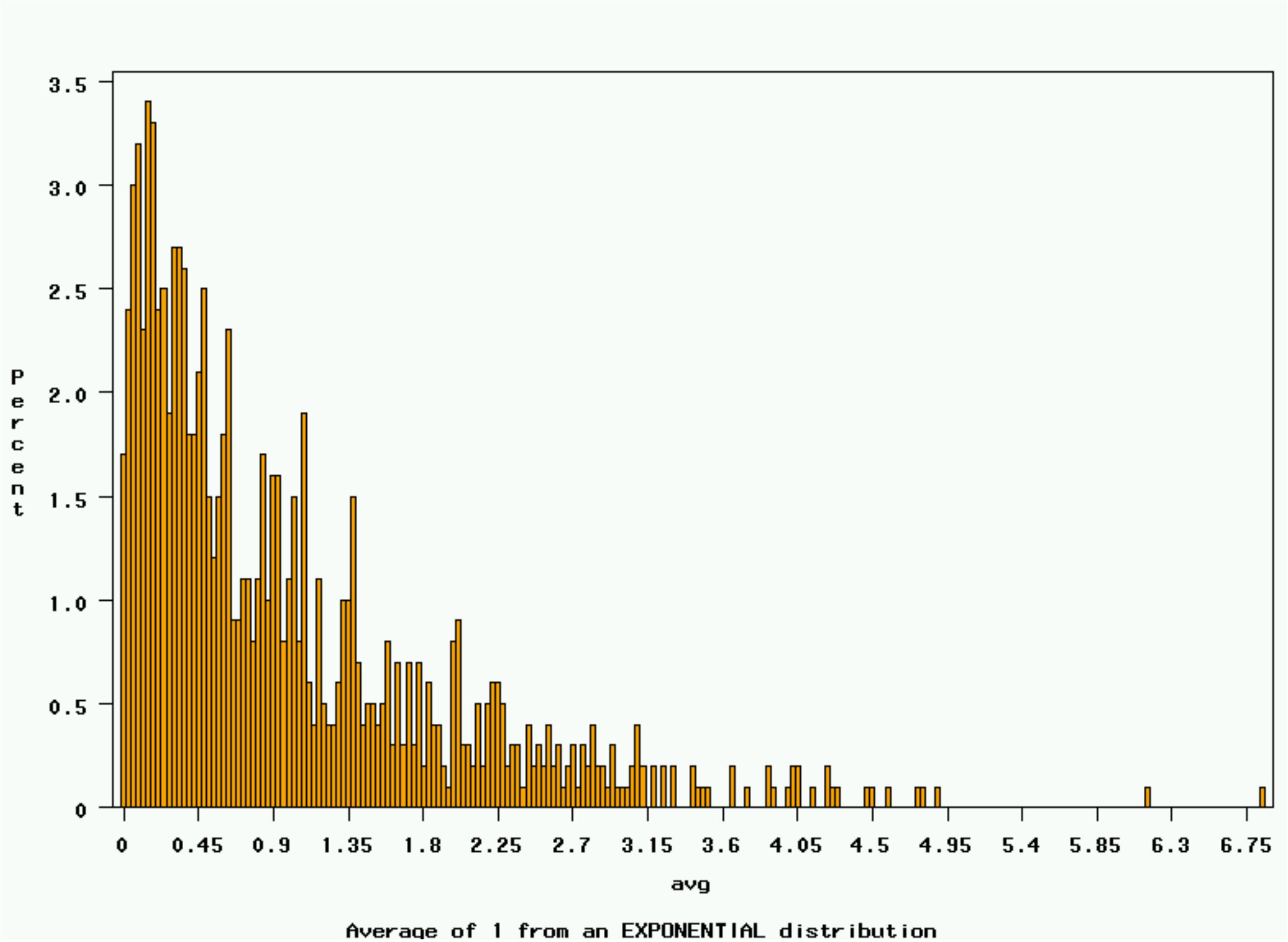
Uniform: 1000 averages of 5



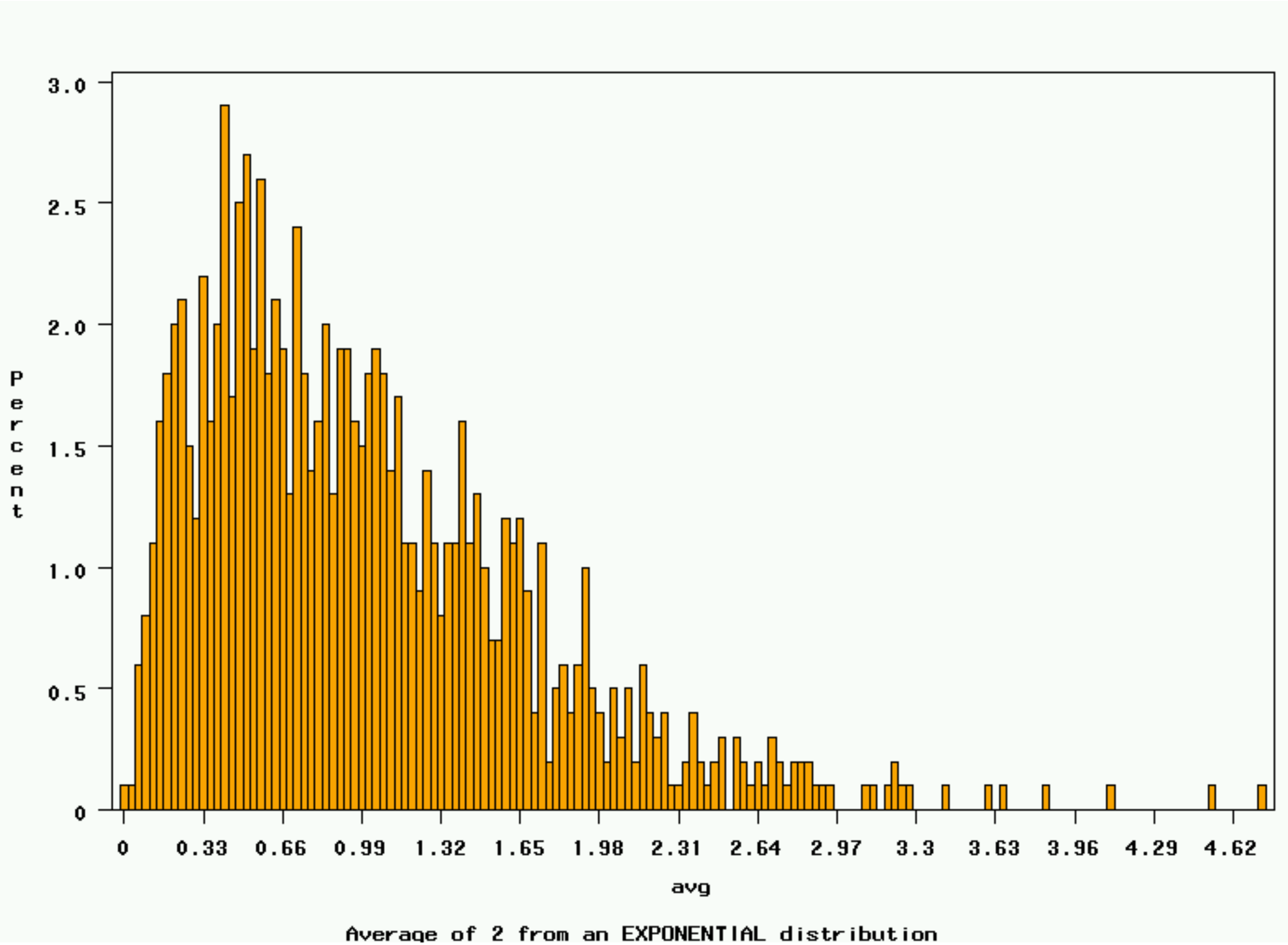
Uniform: 1000 averages of 100



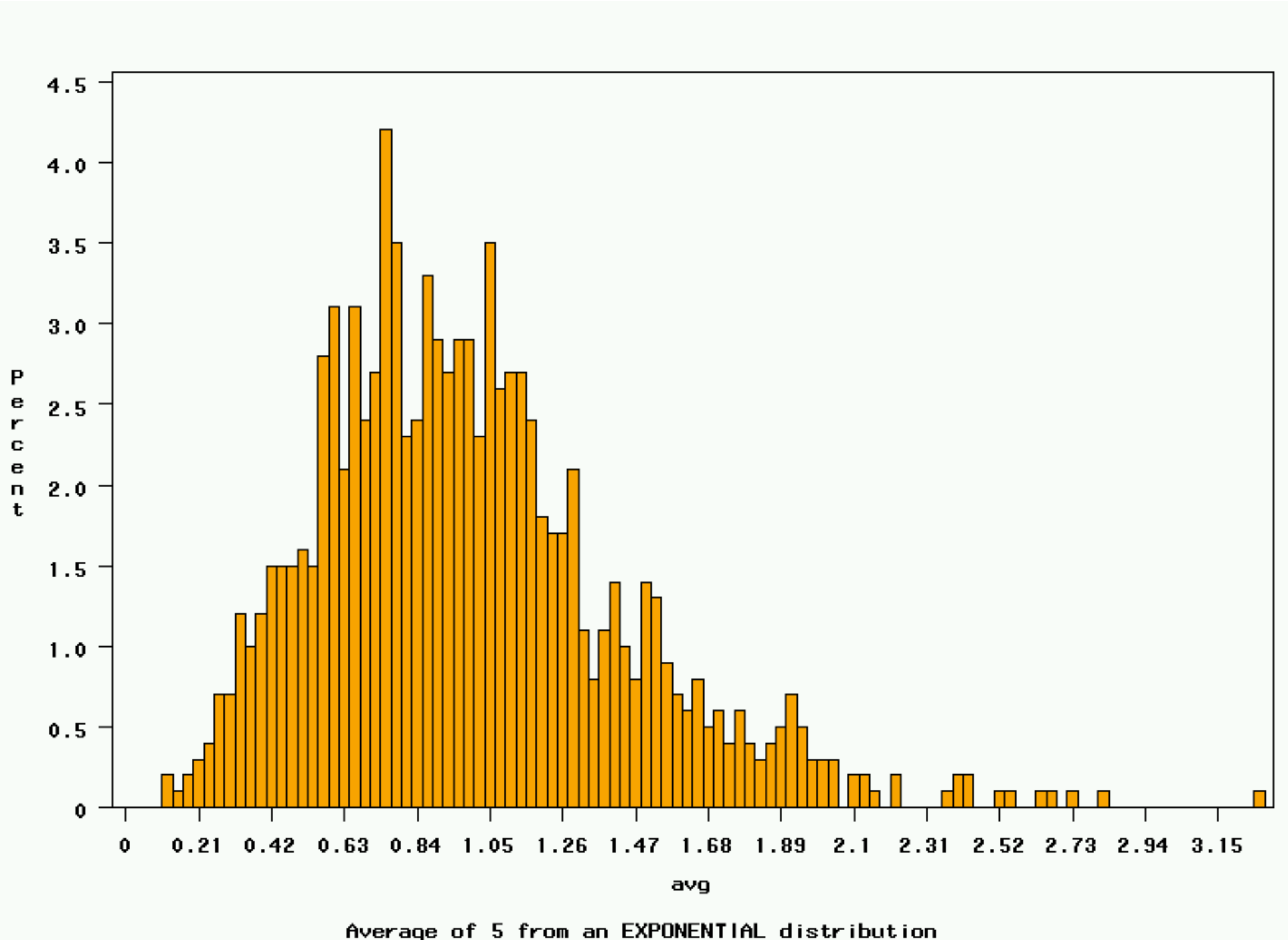
~Exp(1): average of 1 (original distribution)



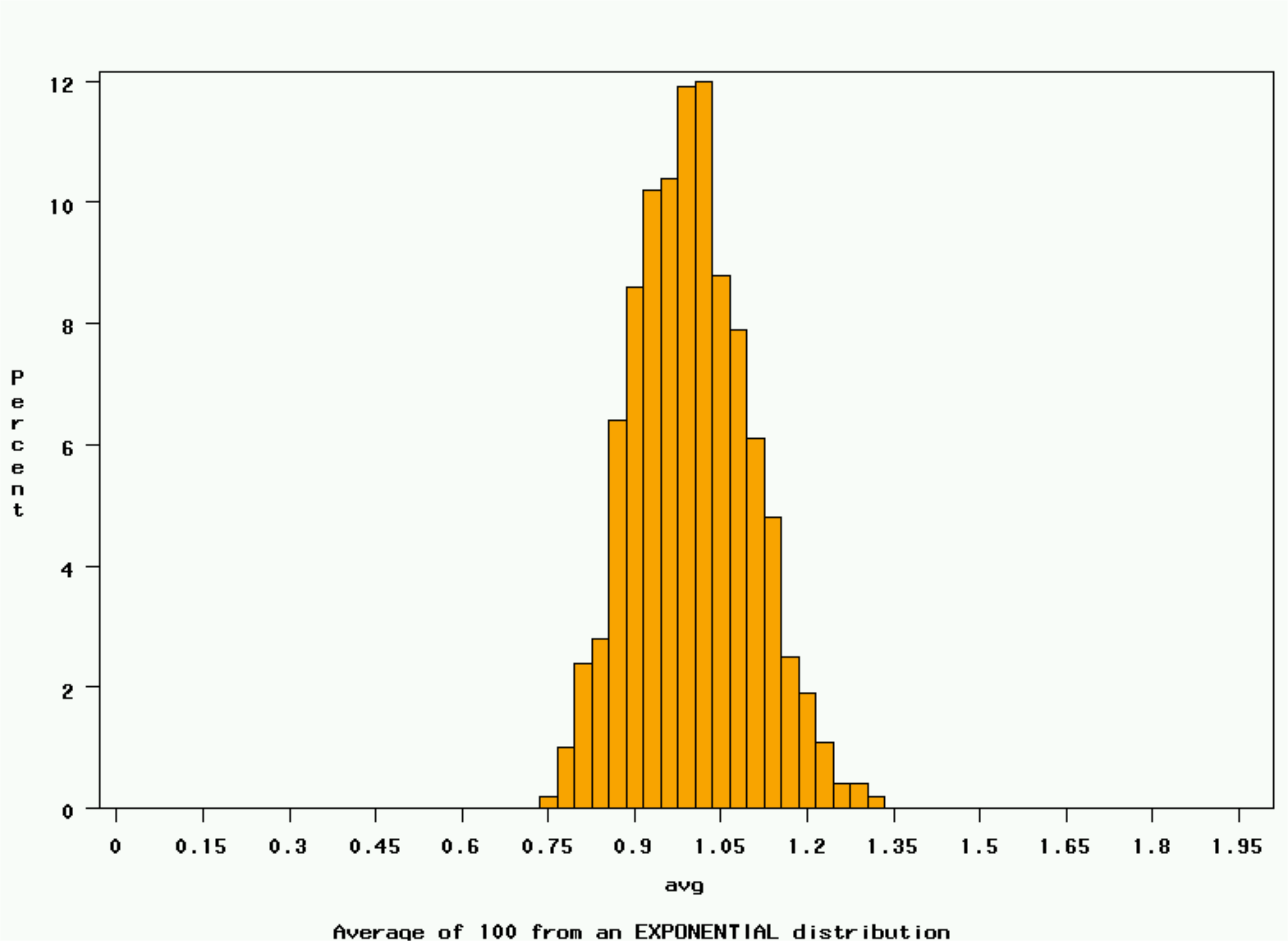
~Exp(1): 1000 averages of 2



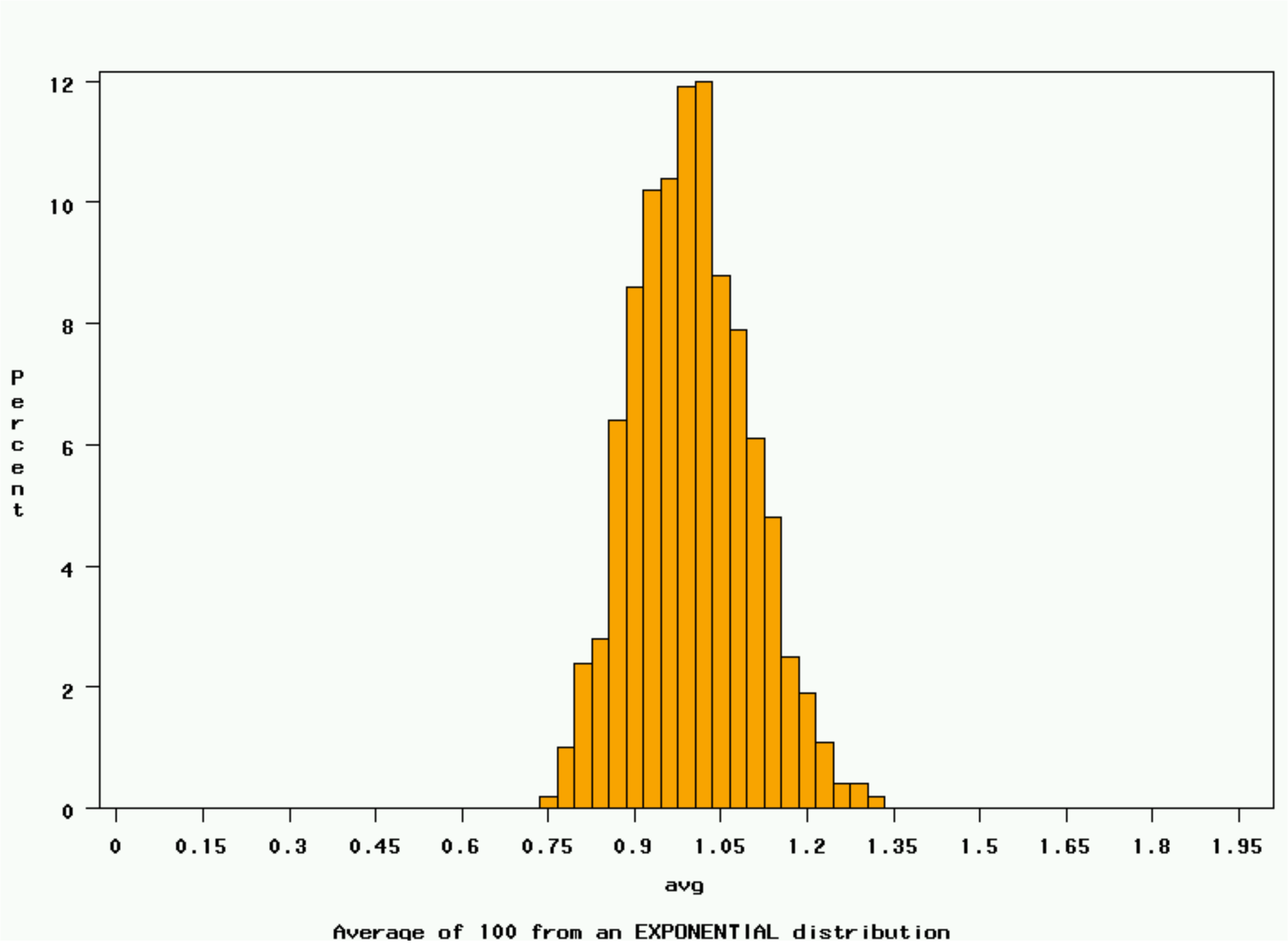
~Exp(1): 1000 averages of 5



~Exp(1): 1000 averages of 100



~Exp(1): 1000 averages of 100



The Central Limit Theorem

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CLT: caveats for small samples

- For small samples:
  - The sample standard deviation is an imprecise estimate of the true standard deviation (σ); this imprecision changes the distribution to a T-distribution.
    - A t-distribution approaches a normal distribution for large n ($\geq$100), but has fatter tails for small n (<100)
  - If the underlying distribution is non-normal, the distribution of the means may be non-normal.

Examples of Sample Statistics

Single population mean

Single population proportion

Difference in means (t-test)

Difference in proportions (Z-test)

Odds ratio/risk ratio

Correlation coefficient

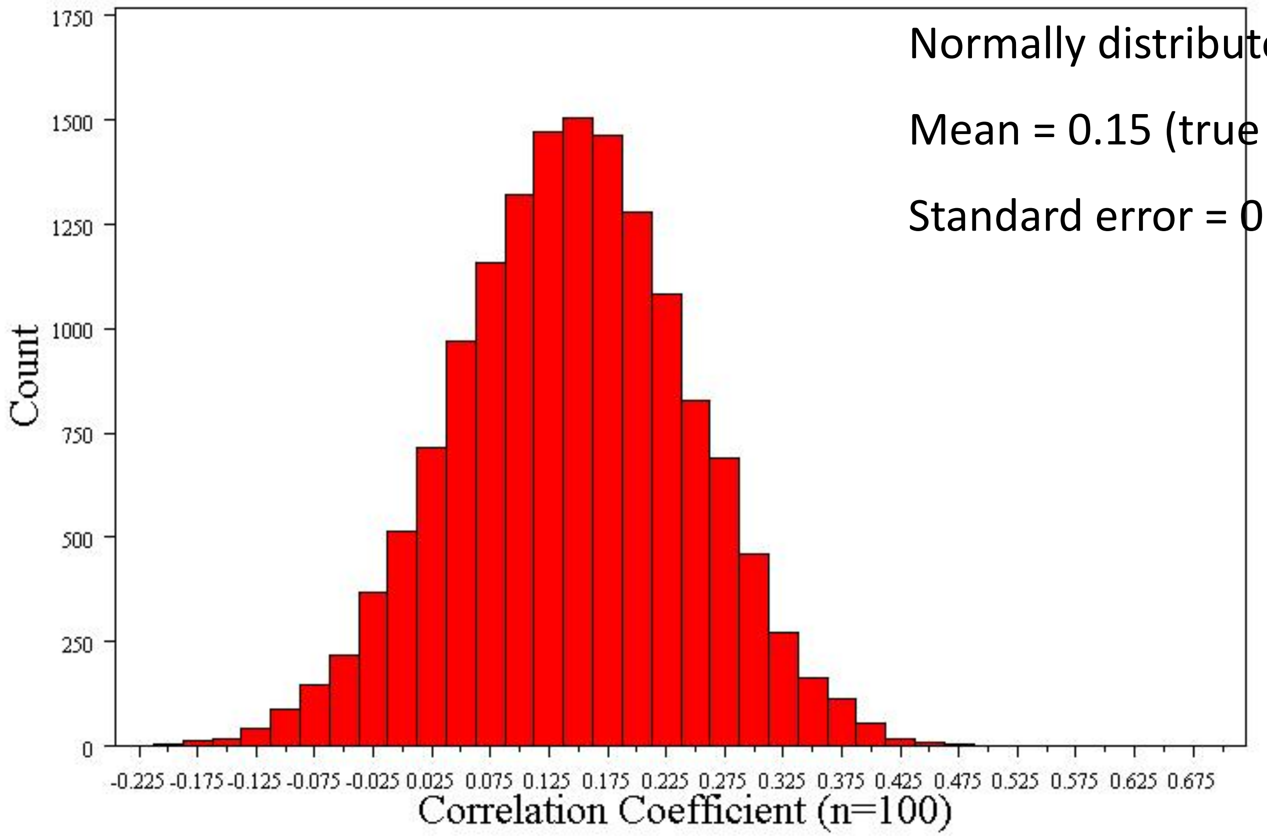
Regression coefficient

…

Distribution of correlation coefficient?

- 1. Specify the true correlation coefficient
  - Correlation coefficient = 0.15
- 2. Select a random sample of 100 virtual men from the population.
- 3. Calculate the correlation coefficient for the sample.
- 4. Repeat steps (2) and (3) 15,000 times
- 5. Explore the distribution of the 15,000 correlation coefficients.

Distribution of correlation coefficient



Normally distributed!

Mean = 0.15 (true correlation)

Standard error = 0.10

Distribution of correlation coefficient

- 1. Shape of the distribution
  - Normally distributed for large samples
  - T-distribution for small samples (n<100)

- 2. Mean = true correlation coefficient (r)

- 3. Standard error $\approx \dfrac{1-r^2}{\sqrt{n}}$

Many statistics follow normal (or t-) distribution

- Means/difference in means
  - T-distribution for small samples
- Proportions/difference in proportions
- Regression coefficients
  - T-distribution for small samples
- Natural log of the odds ratio

3. Confidence Intervals

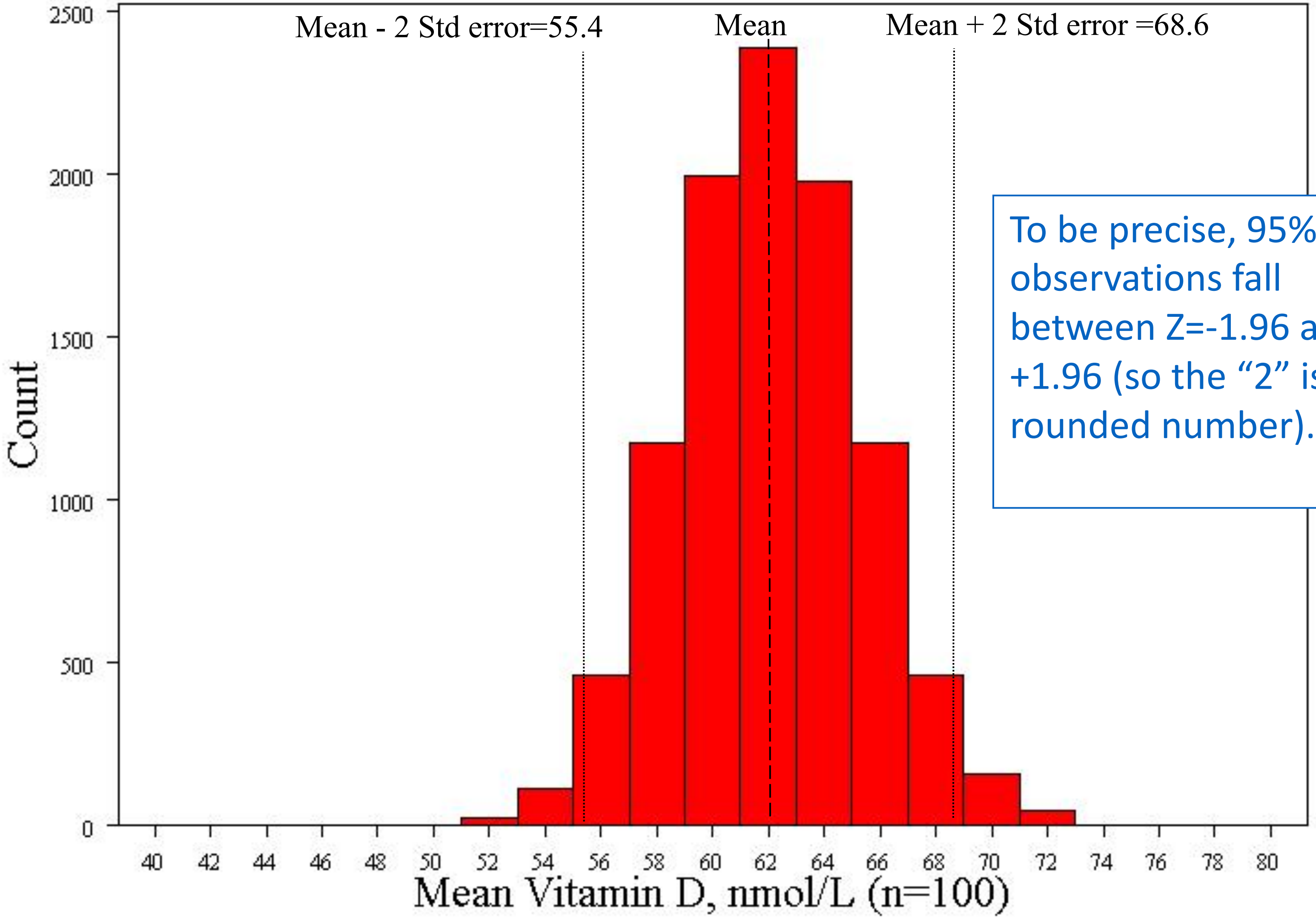
Estimation – confidence intervals

- What is a good estimate for the true mean vitamin D in the population (the population parameter)?
  - 63 nmol/L +/- margin of error

95% confidence interval

- Goal: capture the true effect (e.g., the true mean) most of the time.
- A 95% confidence interval should include the true effect about 95% of the time.
- A 99% confidence interval should include the true effect about 99% of the time.

Recall: 68-95-99.7 rule for normal distributions! These is a 95% chance that the sample mean will fall within two standard errors of the true mean= 62 +/- 2*3.3 = 55.4 nmol/L to 68.6 nmol/L



To be precise, 95% of observations fall between Z=-1.96 and Z= +1.96 (so the “2” is a rounded number)…

95% confidence interval

- There is a 95% chance that the sample mean is between 55.4 nmol/L and 68.6 nmol/L
- For every sample mean in this range, sample mean +/- 2 standard errors will include the true mean:
  - For example, if the sample mean is 68.6 nmol/L:
    - 95% CI = 68.6 +/- 6.6 = 62.0 to 75.2
    - This interval just hits the true mean, 62.0.

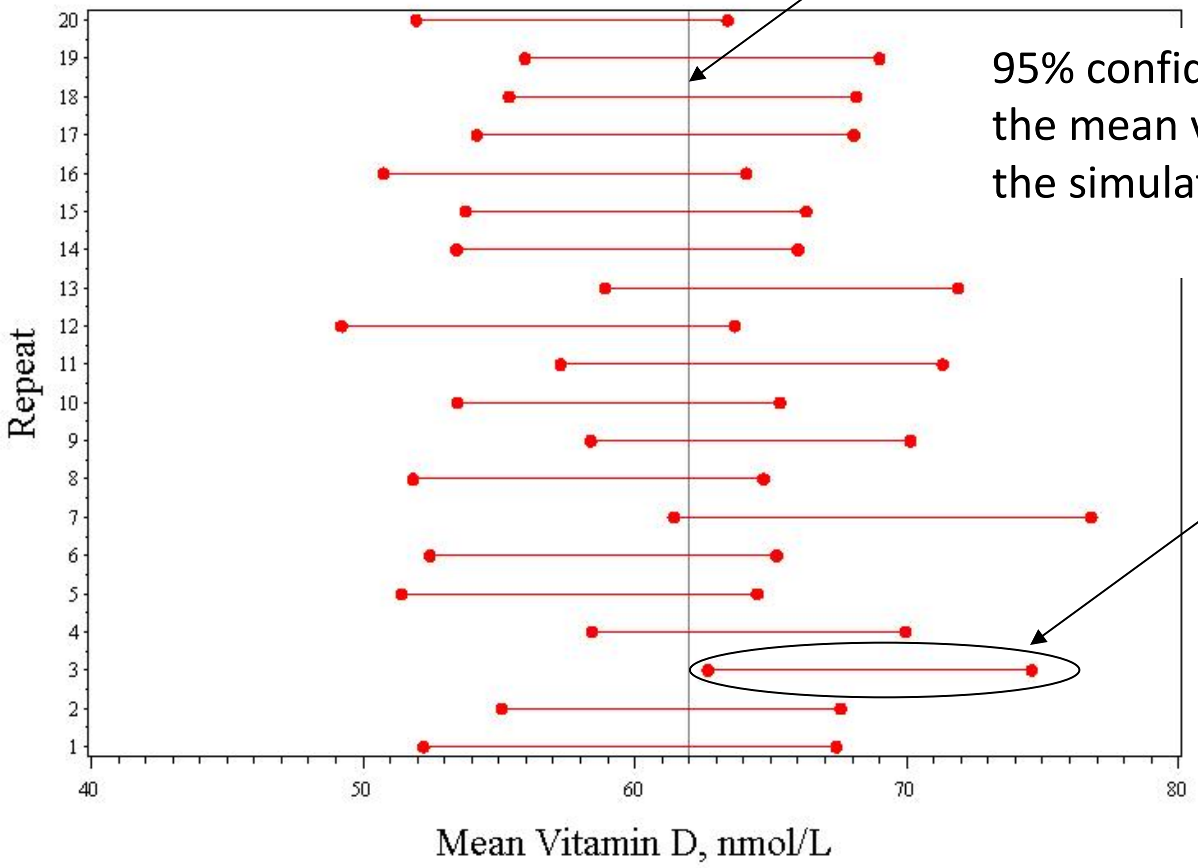
95% confidence interval

- Thus, for normally distributed statistics, the formula for the 95% confidence interval is:
- sample statistic $\pm$ 2 *x* (standard error)
- Examples:
  - 95% CI for mean vitamin D:
    - 63 nmol/L $\pm$ 2 *x* (3.3) = 56.4 – 69.6 nmol/L
  - 95% CI for the correlation coefficient:
    - 0.15 $\pm$ 2 *x* (0.1) = -.05 – .35

Simulation of 20 studies of 100 people

Vertical line indicates the true mean (62)

95% confidence intervals for the mean vitamin D for each of the simulated studies.



Only 1 confidence interval missed the true mean.

Confidence Intervals give:

- A plausible range of values for a population parameter.
- The precision of an estimate.(When sampling variability is high, the confidence interval will be wide to reflect the uncertainty of the observation.)
- Statistical significance (if the 95% CI does not cross the null value, it is significant at .05)

Confidence Intervals

The value of the statistic in my sample (eg., mean, odds ratio, etc.)

point estimate $\pm$ ***(measure of how confident we want to be)*** $\times$ ***(standard error)***

From a Z table or a T table, depending on the sampling distribution of the statistic.

Standard error of the statistic.

Confidence Intervals

The value of the statistic in my sample (eg., mean, odds ratio, etc.)

point estimate* $\pm$ *(measure of how confident we want to be)* $\times$ *(standard error)

From a Z table or a T table, depending on the sampling distribution of the statistic.

Standard error of the statistic.

4. Hypothesis Testing

Hypothesis test

- 1. Is the mean vitamin D in middle-aged and older European men lower than 100 nmol/L (the “desirable” level)?
- 2. Is cognitive function correlated with vitamin D?

Is the mean vitamin D different than 100?

- Start by assuming that the mean = 100
- This is the "null hypothesis"
- This is usually the "straw man" that we want to shoot down
- Determine the distribution of statistics assuming that the null is true…

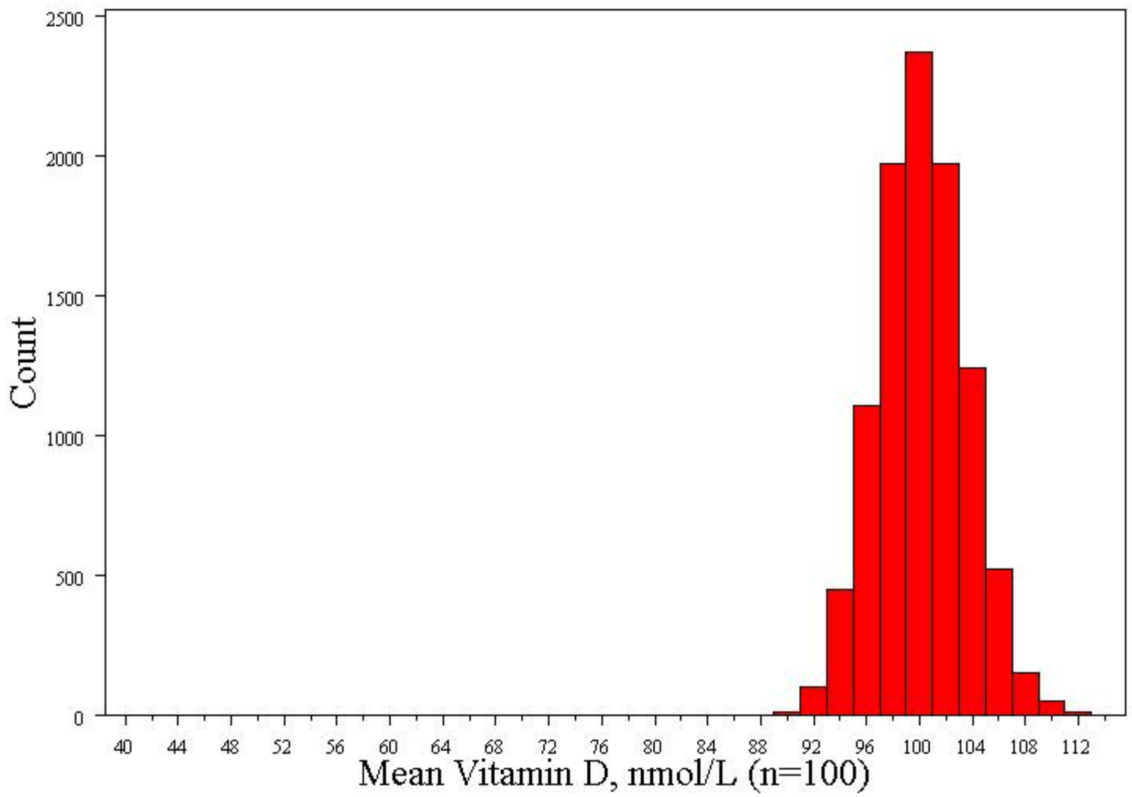
Computer simulation (10,000 repeats)

This is called the null distribution!

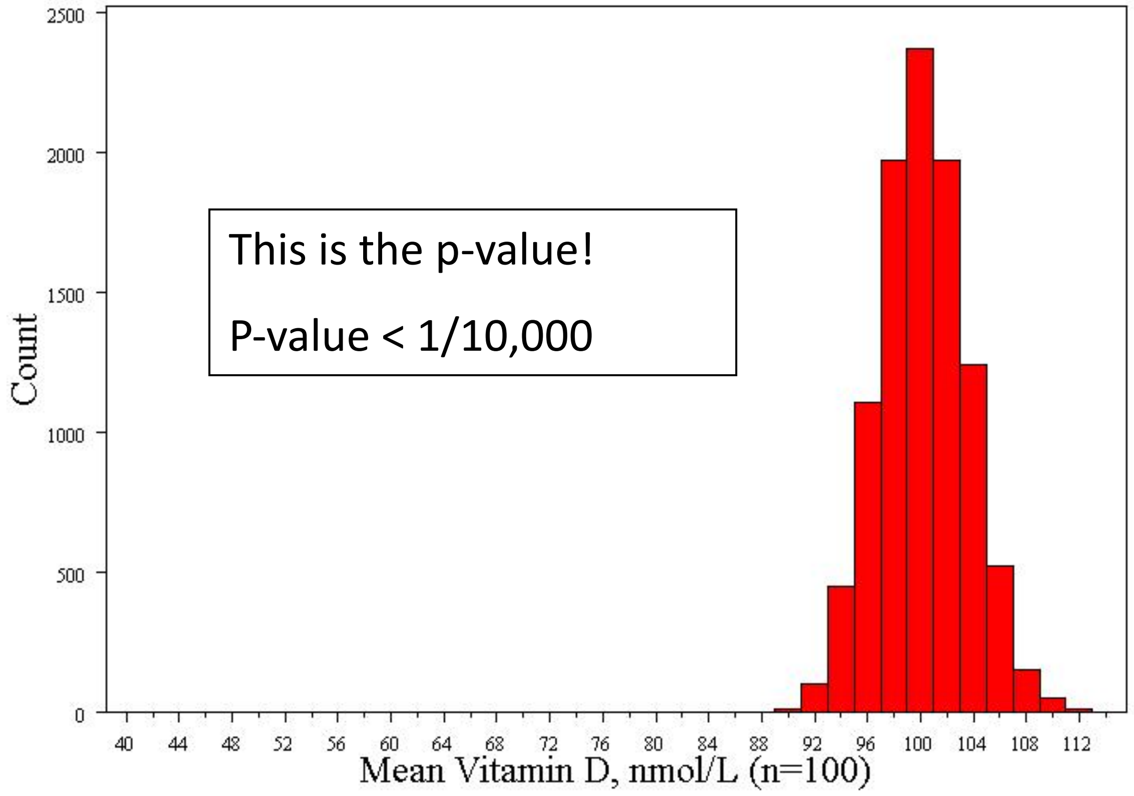
Normally distributed

Std error = 3.3

Mean = 100



Compare the null distribution to the observed value



Calculating the p-value with a formula

Because we know how normal curves work, we can exactly calculate the probability of seeing an average of 63 nmol/L if the true average weight is 100 (i.e., if our null hypothesis is true):

$$\boxed{Z = \frac{63 - 100}{3.3} = 11.2}$$

Z= 11.2, P-value << .0001

The P-value

P-value is the probability that we would have seen our data (or something more unexpected) just by chance if the null hypothesis (null value) is true.

Small p-values mean the null value is unlikely given our data.

Our data are so unlikely given the null hypothesis (<<1/10,000) that I'm going to reject the null hypothesis! (Don't want to reject our data!)

P-value < .0001 means

The probability of seeing what you saw or something more extreme *if the null hypothesis is true (due to chance)*<.0001

P(empirical data/null hypothesis) <.0001

The P-value

By convention, p-values of <.05 are often accepted as “statistically significant” in the medical literature; but this is an arbitrary cut-off.

A cut-off of $p<.05$ means that in about 5 of 100 experiments, a result would appear significant just by chance (“Type I error”).

Summary: Hypothesis testing

<u>The Steps:</u>

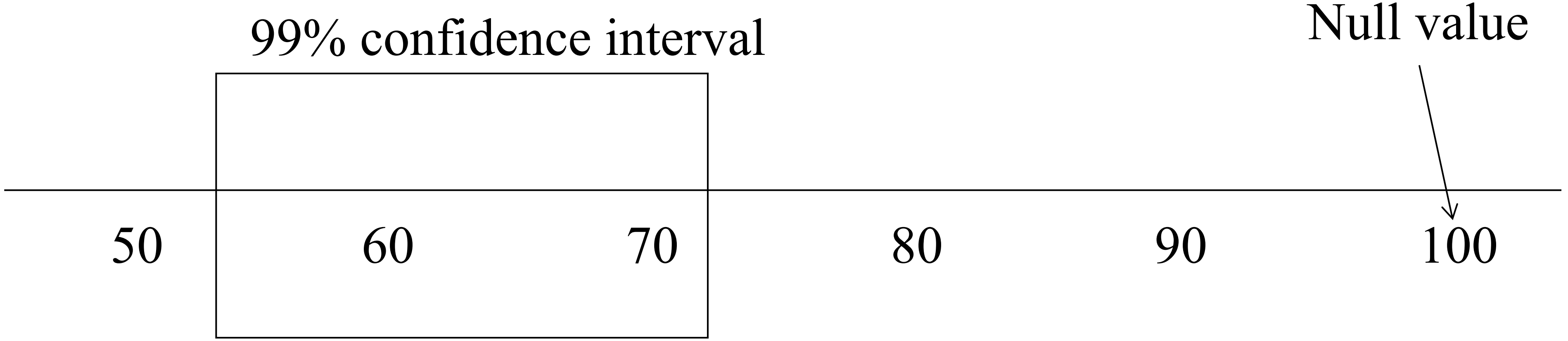
1. Define your hypotheses (null, alternative)
   - ➢ The null hypothesis is the "straw man" that we are trying to shoot down.
   - ➢ Null here: "mean vitamin D level = 100 nmol/L"
   - ➢ Alternative here: "mean vit D < 100 nmol/L" (<u>one-sided</u>)
2. Specify your sampling distribution (under the null)
   - ➢ If we repeated this experiment many, many times, the mean vitamin D would be normally distributed around 100 nmol/L with a standard error of 3.3

$$\boxed{33/\sqrt{100} = 3.3}$$

3. Do a single experiment (observed sample mean = 63 nmol/L)
4. Calculate the p-value of what you observed (p<.0001)
5. Reject or fail to reject the null hypothesis (reject)

Confidence intervals vs hypothesis tests

- Confidence intervals give the same information (and more) than hypothesis tests…
- Duality with hypothesis tests



Null hypothesis: Average vitamin D is 100 nmol/L

Alternative hypothesis: Average vitamin D is not 100 nmol/L (two-sided)

P-value < .01

Is cognitive function correlated with vitamin D?

- Null hypothesis: $r = 0$
- Alternative hypothesis: $r \neq 0$
  - Two-sided hypothesis
  - Doesn't assume that the correlation will be positive or negative.

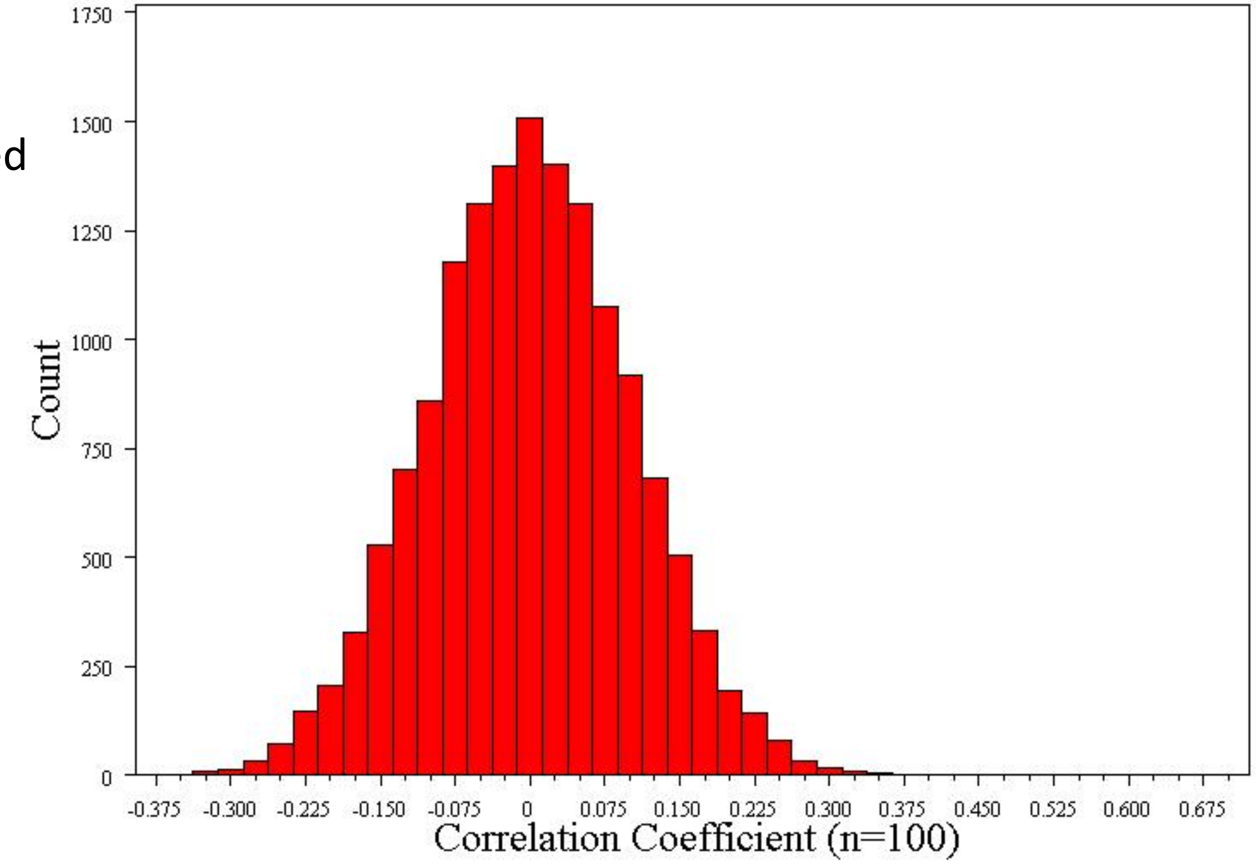
Computer simulation (15,000 repeats)

Null distribution:

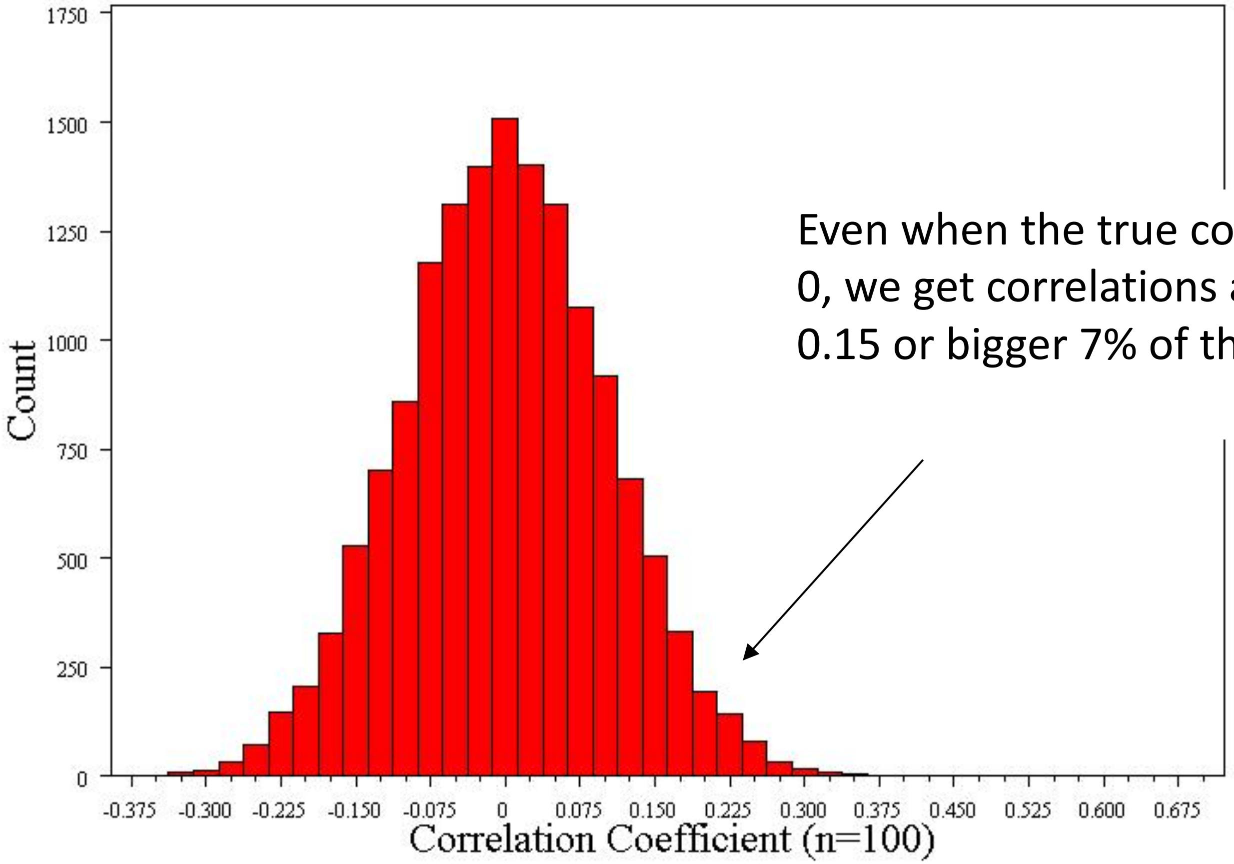
Normally distributed

Std error = 0.1

Mean = 0

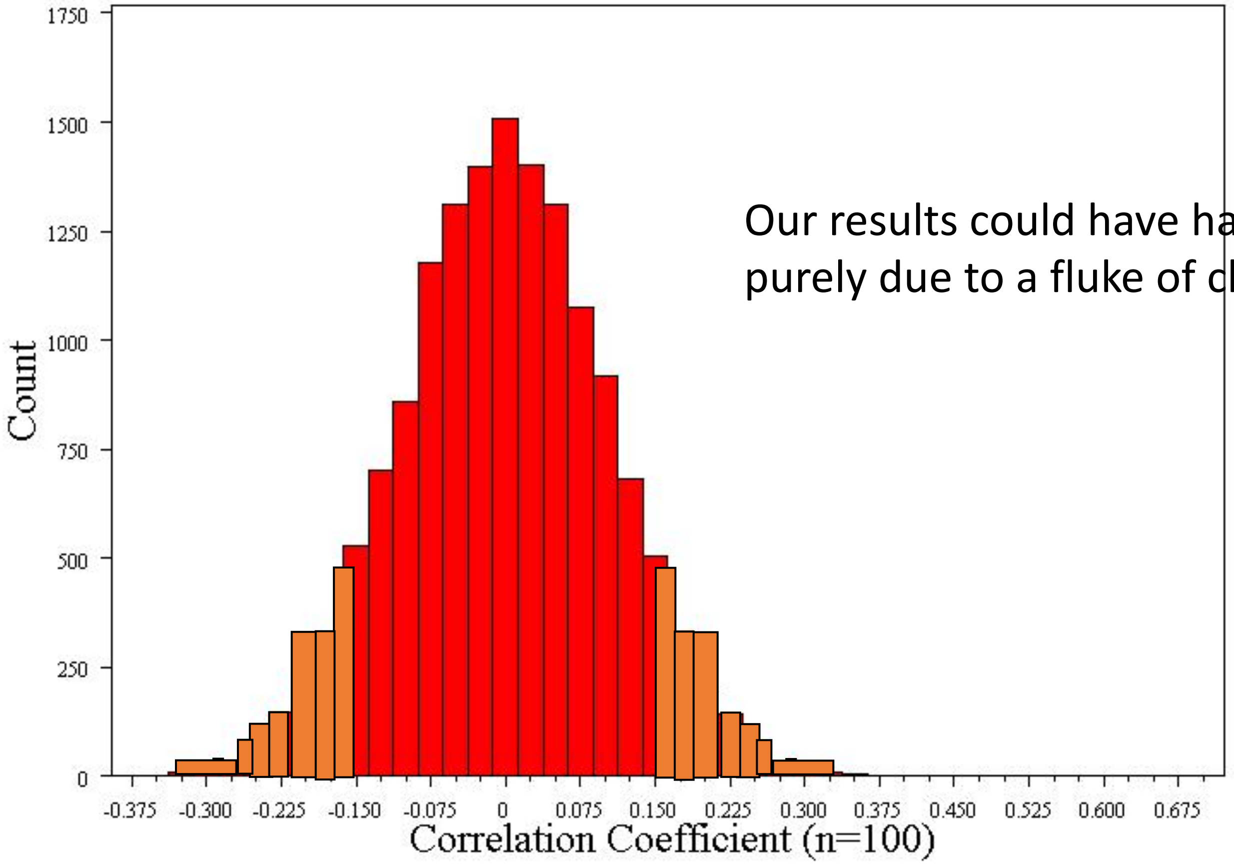


What’s the probability of our data?



Even when the true correlation is 0, we get correlations as big as 0.15 or bigger 7% of the time.

What’s the probability of our data?



Our results could have happened purely due to a fluke of chance!

Formal hypothesis test

- 1. Null hypothesis: $r=0$
  - Alternative: $r \neq 0$ (two-sided)
- 2. Determine the null distribution
  - Normally distributed
  - Standard error = 0.1
- 3. Collect Data, $r=0.15$
- 4. Calculate the p-value for the data:
  - Z =
- 5. Reject or fail to reject the null (fail to reject)

Or use a confidence interval to see statistical significance

- 95% CI = -0.05 to 0.35
- Thus, 0 (the null value) is a plausible value!
- $P>.05$